# Synchronization and coherence in thermodynamic coupled map lattices with intermediate-range coupling

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(Received 23 February 1999)

In spatially extended systems, intermediate-range interactions arise naturally in some physical contexts. To study them, we investigate a model of coupled map lattices (CML's) with intermediate-range coupling, and derive analytic conditions for its synchronization. We find that in these CML's, if the range of coupling is fixed, the law of large numbers applies for the mean field. The total normalized power in nonzero components of the power spectrum of the mean field goes to zero in the thermodynamic limit. We also show that in the same limit the relevant parameter for synchronization and coherence is the fraction of sites coupled, and not their number. [S1063-651X(99)07410-3]

PACS number(s): 05.45.−a

The past decade has seen a surge of interest in synchronization and coherence in coupled map lattices (CML's) [1–10]. Most work has focused on systems with finite degrees of freedom and short-range, global or random coupling. In this paper, we study the synchronization of one- and two-dimensional (1D and 2D) CML's with intermediate-range couplings in the thermodynamic limit. This study is related to systems with finite degrees of freedom studied by Sinha *et al.* [2] and Kozma [10]. In these works, each site is coupled to a range of neighbors on either side, rather than just its nearest neighbors. This may be a valid approximation in systems where interaction strength decays slowly with distance. Just as CML's with nearest neighbor coupling could be considered to be a discretization of partial differential equation, CML's with intermediate-range coupling could be seen as a discretization of a partial integro-differential equation. In fact, such integro-differential equations have often been used in modeling certain physico-chemical reactions [11]. Elimination of rapidly diffusing components in a system of diffusion coupling can also lead to effectively nonlocal coupling in resultant equations [12]. CML's in which coupling between two sites separated by distance $r$ decays as power law $1/r^{\alpha}$ with exponent $\alpha$ have also been considered as models for biological neural networks [6]. In the present work, we derive the analytic conditions for synchronization in our CML's. We find that in the thermodynamic limit the relevant parameter for synchronization is the *fraction* of sites coupled and not their number. We present numerical results which show that for the mean field in the CML's with a fixed range of coupling sites for each site, the law of large numbers applies and the total normalized power in nonzero components of the power spectrum goes to zero in the thermodynamic limit. This is in sharp contrast with the conjecture of Sinha *et al.* [2].

Sinha *et al.* showed that some curious properties emerge in the spatiotemporal dynamics as one increases the range of coupling in a 1D system [2]. In the usual definition of CML's, one couples each site to its nearest neighbors only. Here the coupling is over a finite range, say $B$ neighbors on either side on a lattice of $N$ sites. Sinha *et al.* [2] claimed that the relevant parameter is the number of sites coupled, $B$, and not the fraction of sites coupled, $B/N$. This conjecture was reached on the basis of the observation that power spectra saturate as a function of $B$ and not $B/N$. However, $B$ being a relevant parameter as suggested by Sinha *et al.* [2] is counterintuitive. The intuitive expectation would be that if the system size goes to infinity while the coupling range remains constant, we should obtain results similar to nearest-neighbor coupling at least in one dimension. We show that this expectation is correct and that certain properties which universally emerge in global coupling schemes are absent in this case [8]. The analysis presented by Lemaitre *et al.* [13] also supports our claim. Lemaitre *et al.* showed that coherence can emerge with short-range coupling in dimension $d \geqslant 2$ but not for $d=1$ (see also [14]).

Recently, Kozma [10] numerically studied synchronization in these CML's, and calculated phase diagrams as a function of $B$ and strength of coupling for two different values of $N$. These phase diagrams show striking similarity if the parameter used is $B/N$. In this work, we explicitly prove that at least as far as synchronization is concerned, the relevant parameter is in fact $B/N$ and not $B$. Going to larger lattice sizes, we show that the quasiperiodic behavior that emerges almost universally in globally coupled systems is absent and the law of large numbers applies [1]. We also show that the power spectrum of mean field tends to a $\delta$ function with the peak at zero momentum in the thermodynamic limit. Thus the structure seen by Sinha *et al.* disappears in this limit.

Let us first consider a linear lattice of $N$ sites with periodic boundary conditions. We assign a real variable $x_i(t)$ at each site $i$, $i=1,\ldots,N$. Let $B=N/k$ be the number of neighbors on either side for each site. Each site evolves as follows:

$$x_i(t+1)=(2B)^{-1}\sum_{j=1}^{B}\left[f(x_{i+j}(t))+f(x_{i-j}(t))\right]. \qquad (1)$$

The Jacobian of the above system at any time has elements

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$j_{m,l}(t)=(2B)^{-1}f'(x_l(t))$ if (a) $0<|m-l|\leqslant B$, or (b) $|m+N-l|\leqslant B, m\leqslant B$, or (c) $|N-m+l|\leqslant B, m>N-B$; the other elements are zero. We assume that $N>2B+1$ to avoid complications. It is clear that synchronized state, i.e., the state in which $x_i(t)=x(t)\forall i$ for all times $t$, is a solution of the system. The reason is that if one starts in a synchronized state, the system stays synchronized. Let us check the stability of this state. In this state the Jacobian matrix is related to the interaction matrix by $j(t)=If'(x(t))$, where $I$ is an $N$-dimensional interaction matrix with elements such that $I_{m,l}=1/(2B)$ if conditions (a), (b), or (c) defined above are satisfied; the other elements are zero. The long term Jacobian for the synchronized state is given by $J_t=j(t)\cdots j(2)j(1)=I^t f'(x_t)\cdots f'(x_2)f'(x_1)$. Thus the stability of this state depends on the eigenvalues of the interaction matrix and the Lyapunov exponent of the map $f$. Let $\lambda_i$, $i=0,\ldots,N-1$ denote $N$ eigenvalues of $I$ and $\lambda$ be the Lyapunov exponent of the map $f$. In order to find the eigenspectrum of the interaction matrix $I$, the symmetries are helpful. The interaction matrix is a circulant matrix [3,15]. The Fourier modes are the eigenmodes of the interaction matrix. Thus the stability of any synchronized state can be analyzed by expanding perturbations in terms of Fourier modes. We will analyze synchronous chaotic state which is widely observed in these systems [10]. The only eigenmode which corresponds to the uniform state is one for $q=0$, i.e., $[1,1,\ldots,1]$. It is easy to show that the condition for the stability for synchronous chaos is that only this eigenmode should survive and the rest should be damped [4,5]. Let $\lambda_0$ be the eigenvalue corresponding to this eigenmode with $q=0$ and $\lambda$ be the Lyapunov exponent of the map $f$. The necessary condition for synchronous chaos is that only one eigenvalue $|\lambda_0 e^{\lambda}|>1$, with $|\lambda_i e^{\lambda}|<1$ for $i=1,\ldots,N-1$ [16].

Let us compute the eigenvalues of interaction matrix. Using the symmetries of interaction matrix, we find

$$\lambda_l=B^{-1}\sum_{j=1}^{B}\cos(2jl\pi/N), \tag{2}$$

where $l=0,\ldots,N-1$. Thus $\lambda_0=1$ and $\lambda_l<\lambda_0$ for $l\neq 0$. The identity $\Sigma_{j=0}^{B}\cos(j\theta)=\frac{1}{2}[1+\sin((B+\frac{1}{2})\theta)/\sin(\theta/2)]$ implies that $\lambda_l=1/B[1/2+\sin(2\pi(B+\frac{1}{2})l/N)/2\sin(\pi l/N)]-(1/B)$, for $l\geqslant 1$. The $\lambda_l$'s are continuous functions of $l$, and $\lambda_1$ can be arbitrarily close to $\lambda_0$. Let us determine the bounds of stability in terms of $\lambda_1$ alone. For large $N$, $(\sin(\pi/N)\approx\pi/N)$ and for $B\gg\frac{1}{2}$ we obtain

$$\lambda_1\sim\frac{\sin(2\pi/k)}{2\pi/k}. \tag{3}$$

Thus the eigenvalue $\lambda_1$ is distinct from $\lambda_0=1$, and this gap in the spectrum can be used to establish synchronization for constant $k$ even when $N\to\infty$. (The bounds for $\lambda_l$'s for $l\geqslant 1$ can be found, and it can be shown that $\lambda_1$ remains the value with largest modulus for any $k>2$.) In other words, it is possible to find the function with the Lyapunov exponent $\lambda$, such that $e^{\lambda}>1$ but $|\lambda_1 e^{\lambda}|<1$ for some value of $k$. However, the situation changes qualitatively if $B$, number of neighbors at each site, remains constant as considered by Sinha *et al.* [2]. In this case, as $N\to\infty$, $k\to\infty$, $2\pi/k\to 0$, and $\lambda_1\to 1$.

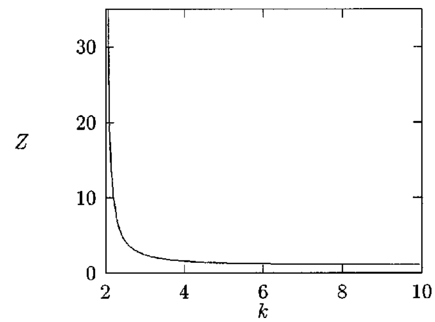


FIG. 1. This figure shows the behavior of the maximum allowed value of $Z=e^{\lambda}$ as a function of fraction $k$ of sites coupled.

This implies that synchronous chaos is not possible in the thermodynamic limit. Thus, in the thermodynamic limit, the system behaves like a nearest-neighbor coupled system. This is what one would intuitively expect. Let us take another extreme case, that of global coupling $B=N/2$ $(k=2)$. One should always see a synchronized state in this case. It is easy to find the range of $e^{\lambda}<Z$ of a single map in which synchronized chaos will be observed. For $k=5/2$, $Z=4.27\ldots$; for $k=3$, $Z=2.418\ldots$; and for $k=6$, $Z=1.20\ldots$. The stability range approaches unity for large $k$. In Fig. 1, we show the dependence of $Z$ on $k$. We have verified these results in numerical simulations.

Let us consider a case with on-site contribution along with $B$ nearest neighbors, i.e.,

$$x_i(t+1)=(1-\epsilon)f(x_i(t))+(\epsilon/2B)\sum_{j=1}^{B}[f(x_{i+j}(t))+f(x_{i-j}(t))]. \tag{4}$$

The stability analysis can be done on similar lines. The eigenvalues change as $\lambda_i\to(1-\epsilon)+\epsilon\lambda_i$. However, the behavior in the thermodynamic limit does not change.

These conclusions for synchronization remain unchanged even if one considers higher dimensional CML's with intermediate-range interactions. Let us consider the evolution of a 2-d $L\times L$ lattice of $N$ sites with periodic boundary conditions, where $N=L^2$:

$$x_{i,j}(t+1)=(4B)^{-1}\sum_{k=1}^{B}[f(x_{i+k,j}(t))+f(x_{i-k,j}(t))+f(x_{i,j+k}(t))+f(x_{i,j-k}(t))].$$

The interaction matrix of this system is a block-circulant matrix with circulant blocks. Using Ref. [3], we find that the $L^2$ eigenvalues are

$$\lambda_{l,m}=(2B)^{-1}\sum_{j=1}^{B}(\cos(2jl\pi/L)+\cos(2jm\pi/L)),$$

where $l=0,\ldots,L-1$, and $m=0,\ldots,L-1$. Thus $\lambda_{0,0}=1$ and $\lambda_{l,m}<\lambda_{0,0}$ for nonzero $l$ and $m$. The crucial factor in synchronization is how close $\lambda_{0,0}$ is to $\lambda_{0,1}=\lambda_{1,0}$. Using arguments as in the 1D case, one can show that $\lambda_{1,0}\sim 1/2[1+\sin(2\pi/k)/(2\pi/k)]$. As in the 1D case, synchronization is impossible for constant $B$ in the thermodynamic limit. The synchronization condition remains the same.

Thus it is clear that synchronization cannot emerge with short interactions on Euclidean lattices in general. However, one can still ask if some other kind of collective behavior can develop. Sinha *et al.* [2] concluded that coupling with a range larger than some critical length is similar to global coupling, as a result of their numerical simulations. Using an extra array of partial sums, we can simulate much larger lattices. We show that (a) unlike the case of globally coupled maps, the law of large numbers is applicable, and (b) the power spectrum of the mean field does not have any particular structure in the thermodynamic limit.

In CML's with global coupling it was found that the fluctuations in the mean field do not decay as $1/N$ even though all of the Lyapunov exponents were found to be positive and there was no apparent order in the power spectrum [1]. It was found that the fluctuations as defined below saturate, signifying subtle collective behavior. In particular, if one defines the mean field as $h(t)=N^{-1}\Sigma_{i=1}^{N}f(x_i(t))$ then its standard deviation $\sigma^2=(\langle h^2\rangle)-(\langle h\rangle^2)$ would decay as $1/N$ if the $f(x(i))$'s were independent random numbers. Of course, the sites in CML's are not independent. However, in one dimension, if the lattice is spatially and temporally uncorrelated, i.e., chaotic and not synchronized, one can see that the law of large numbers is applicable. The reason for this is that in such a situation the correlation decays exponentially and the sites beyond the correlation length are effectively independent. This is why dimension and similar quantities are extensive in this case [17]. We expect the correlation length of intermediately coupled CML's not to change if one fixes the number of neighbors $B$. We simulated Eq. (4) for $B=500$, at which the variance in the mean field of globally coupled maps is near the asymptotic value [1]; Sinha *et al.* [2] used the same value of $B$ in their simulations. However, we see no signs of saturation in our simulations of intermediate-range coupling. This is expected if correlation length a function of $B$ alone. Figure 2(a) shows that variance $\sigma^2$ as a function of $N$. The law of large numbers clearly holds.

Given that the deviations around mean field decay as $1/N$ in this scheme, one would expect that the contribution to the zeroth component in the power spectrum of $h(t)$ would keep increasing at the cost of other components. This is because the zeroth component represents the mean. Figure 2(b) shows the fraction of power in nonzero components as a function of $N$. We find that the total normalized power in nonzero components of the power spectrum decreases continually as $N$ increases. Thus, in the thermodynamic limit, the power spectrum tends to a $\delta$ function in the zeroth component as expected. One does not expect any long range order in one dimension with short-range interaction. The structure observed by Sinha *et al.* [2] in the power spectrum disappears in the thermodynamic limit.

CML's with power-law couplings can sometimes be expected to have behavior similar to CML's with short-range couplings. Even a $1/r^2$ interaction does not produce synchronization [6]. [A proof can be given with the formalism used above and $\Sigma_{i=1}^{N}i^{-2}\cos(2\pi i/N)\rightarrow\Sigma_{i=1}^{N}i^{-2}$ for large $N$.] We have checked the mean square deviation of the mean field as a function of $N$ for unsynchronized CML's in which the couplings decay as $I_{i,j}=r_{ij}^{-\alpha}/(2\Sigma_{k=1}^{N/2}k^{-\alpha})$ where $r_{i,j}$ $=|i-j|$ if $|i-j|<N/2$ and $r_{i,j}=N-|i-j|$ if $|i-j|>N/2$ for $\alpha=1$ and 2. Unfortunately, it is not possible to go to very large system sizes in this case since we did not find any technique to increase the speed of numerical simulations. However, we found that the fluctuations do not decay as $1/N$ (but some anomalous power) when couplings decay as $1/r$, and they do decay as $1/N$ when couplings decay as $1/r^2$. These results are shown in Fig. 3. It seems that the couplings in which synchronization is possible also give rise to large-scale coherences. This is reasonable since synchronization is an extreme case of coherence. (If we assume that correlations

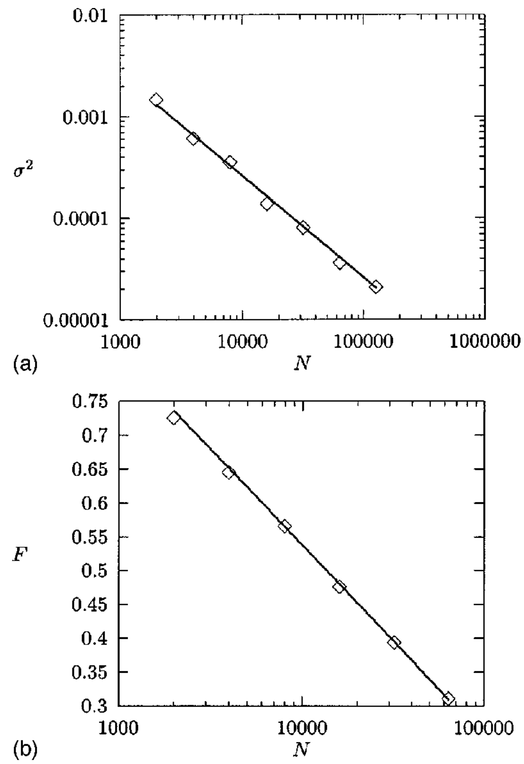


FIG. 2. Data for the map of Eq. (4) on a lattice of $N$ sites with $f(x)=1-1.99x^2$, $B=500$, and $\epsilon=0.1$. (a) The variance of the mean field, $\sigma^2$, as a function of $N$ on a log scale. (b) Fraction of power in nonzero components of the power spectrum, $F$, as a function of $N$ on a semilog scale.

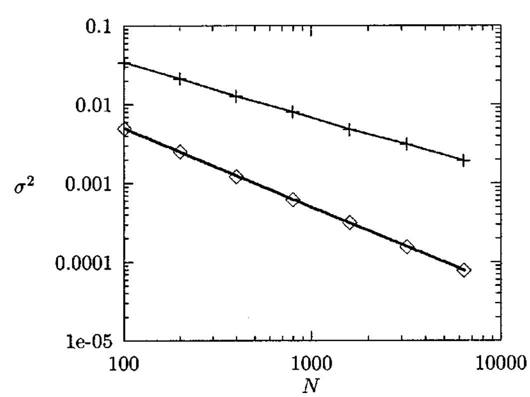


FIG. 3. The variance of the mean field, $\sigma^2$, as a function of $N$ on a log scale for the coupling with $I_{i,j}=(1/r_{ij}^{\alpha})/[2\Sigma_{k=1}^{N/2}(1/k^{\alpha})]$, $f(x)=g^3(x)$, $g(x)=1-ax^2$, and $a=1.99$. Strong chaos is chosen to avoid synchronization. The upper line shows data for $\alpha=1$, and the lower line shows data for $\alpha=2$.

in the chaotic case decay as fast as couplings, then a system whose couplings decay as $1/r^2$ or faster would have finite correlation length independent of system size. Thus the parts of system larger than the correlation length can be considered independent. This could be the reason for different behaviors at different values of $\alpha$.) A detailed analysis of such systems is being pursued.

We would like to point out here that in 1D cellular automata, one can rigorously formulate a local structure theory. In this theory, it is possible to decompose the probabilities of blocks into that of subblocks due to the shift invariant nature of evolution rules [18]. In CML's, given their shift invariant nature, a similar theory should be possible. However, one should note that such a rigorous decomposition in blocks of arbitrary size is not possible in higher dimensions, even in principle. According to Ref. [18] this is related to the undecidability of whether a plane can be tessellated with a given collection of polygons. In one dimension, if the correlation length is finite, blocks of size larger than the correlation length should be effectively independent. Since the subblocks determine the bigger block, nontrivial collective behavior is not expected to emerge in one dimension for a large system.

In short, intermediate-range coupling with fixed $B$ is not qualitatively different from nearest neighbor coupling in the thermodynamic limit. Thus studies on intermediate-range coupling with fixed $B$ are unlikely to yield any new understanding. However, experimentally, the infinite lattice limit is not realistic and for coupling ranges comparable to system size, such studies can be useful.

We would like to thank J. G. Dushoff for a critical reading of the paper. This work was supported in part by the National Science Council of the Republic of China (Taiwan) under Grant No. NSC 88-2112-M-001-011.

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